

Quantum Phase Transition of Condensed Bosons in Optical Lattices

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In this paper we study the superfluid-Mott-insulator phase transition of ultracold dilute gas of bosonic atoms in an optical lattice by means of Green function method and Bogliubov transformation as well. The superfluid-Mott insulator phase transition condition is determined by the energy-band structure with an obvious interpretation of the transition mechanism. Moreover the superfluid phase is explained explicitly from the energy spectrum derived in terms of Bogliubov approach.

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I. INTRODUCTION

Ultracold bosons in an optical lattice provide a tunable quantum system with variance of the potential depth and lattice constant which can be achieved by adjusting the parameters of the configuration of laser beams. Various quantum phenomena, for instance, Bloch oscillations, Wannier-Stack ladders [1] have been investigated in such a system which shares spatial periodicity with crystal lattice in solid state physics, however is immune from scattering of impurities or phonons. The superfluid-Mott-insulator (SMI) phase transition is one of the most significant quantum phenomena of condensate bosons in the optical lat-

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tice. A known analogous system exhibiting the SMI phase transition is liquid helium with short-range repulsive interaction in periodic potential [8]. The atomic gas in Bose-Einstein condensate (BEC) subjected to the lattice potential which is turned on smoothly can be kept in the superfluid phase (SFP) as long as the repulsive interaction between atoms is small with respect to the tunnel coupling. With increasing of the potential depth of the optical lattice it is getting more and more difficult for bosons to tunnel from one site to the other, and finally the system attends an insulator phase above a critical value of the potential depth. Considerable attention has been attracted to theoretical researches for understanding of the phase transition and determining of the transition condition [2]- [7], in which the Bose-Hubbard Model(BHM) is introduced as the starting point of the theoretical studies [2]. The phase transition phenomena have been also observed experimentally in BEC loaded in a three dimensional optical lattice [2]. Using a strong coupling expansion in terms of the hopping term called the decoupling approximation, which is as a matter of fact based on the mean-field method D. van Oosten et. al have obtained an analytic phase transition condition (see the following Eq.(27)). From an alternative viewpoint the phase transition condition can be also determined from the energy spectrum of the system since the excitation spectrum is necessarily gapless for the SFP while has a finite gap for the Mott insulator phase (MIP). We in the present paper use both the Green function method and Bogliubov approach to obtain the explicit excitation energy spectrum and hence to investigate the SMI phase transition.

We begin with the following second-quantized Hamiltonian operator [2] for the system of bosonic atoms in the optical lattice,

$$\begin{aligned} \hat{H} = \int d\mathbf{x} \hat{\psi}^\dagger(\mathbf{x}) \left(-\frac{\hbar^2}{2m} \nabla^2 + V_0(\mathbf{x}) + V_T(\mathbf{x}) \right) \hat{\psi}(\mathbf{x}) \\ + \frac{g}{2} \int d\mathbf{x} \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{x}) \hat{\psi}(\mathbf{x}) \end{aligned} \quad (1)$$

where $\hat{\psi}(\mathbf{x})$ and $\hat{\psi}^\dagger(\mathbf{x})$ denote the boson field operators which obey the boson commutation relation

$$\left[\hat{\psi}(\mathbf{x}), \hat{\psi}^\dagger(\mathbf{x}') \right] = \delta(\mathbf{x} - \mathbf{x}'). \quad (2)$$

Here $V_0(\mathbf{x}) = \sum_{j=1}^3 V_{j0}(\mathbf{x}) \sin^2(\frac{2\pi x_j}{\lambda})$ is the potential of the optical lattice formed by the laser light of wavelength λ , and hence the lattice constant is $d = \lambda/2$. $V_T(\mathbf{x})$ denotes an external trap potential, and the interparticle interaction is approximated by the short-range potential $g\delta(\mathbf{x} - \mathbf{x}')$, where $g = 4\pi a_s \hbar^2/m$ is the coupling constant with a_s the s-wave scattering length. Expanding the field operator $\hat{\psi}(\mathbf{x})$ in the Wannier basis such that $\hat{\psi}(\mathbf{x}) = \sum_i \hat{a}_i w(\mathbf{x} - \mathbf{x}_i)$, we obtain the Bose-Hubbard model

$$\hat{H} = -J \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_j + \sum_i \varepsilon_i \hat{n}_i + \frac{1}{2} U \sum_i \hat{n}_i (\hat{n}_i - 1) \quad (3)$$

where \hat{a}_i is the annihilation operator of a particle at the lattice site i , which is assumed as being in a state described by the Wannier function $w(\mathbf{x} - \mathbf{x}_i)$ of the lowest energy band localized on i th site. This implies the assumption that the energy involved in the system is small comparing with the excitation energies of the second band. \mathbf{x}_i denotes the position of the i th local minimum of the optical potential, and $\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$ is the number operator. The annihilation and creation operators \hat{a}_i and \hat{a}_i^\dagger obey the canonical commutation relations $[\hat{a}_i, \hat{a}_i^\dagger] = \delta_{ij}$. The parameter J is the hopping matrix element between adjacent sites i, j , and is evaluated as

$$J = - \int d\mathbf{x} w^*(\mathbf{x} - \mathbf{x}_i) \left[-\frac{\hbar^2}{2m} \nabla^2 + V_0(\mathbf{x}) \right] w(\mathbf{x} - \mathbf{x}_j). \quad (4)$$

The energy offset of each lattice site, $\varepsilon_i = \int d\mathbf{x} V_T(\mathbf{x}) |w(\mathbf{x} - \mathbf{x}_i)|^2 \approx V_T(\mathbf{x}_i)$, is assumed to be of the same value ε in the present paper. The interparticle interaction is characterized by the parameter

$$U = g \int d\mathbf{x} |w(\mathbf{x})|^4 \quad (5)$$

For the sake of convenience we rewrite the Hamiltonian(3) in the following form:

$$\hat{H} = \hat{H}_0 + \frac{1}{2} U \sum_i \hat{n}_i (\hat{n}_i - 1), \quad \hat{H}_0 = \sum_{i,j} T_{ij} \hat{a}_i^\dagger \hat{a}_j \quad (6)$$

where

$$T_{ij} = \begin{cases} \varepsilon & \text{for } i = j \\ -J & i, j \text{ are nearest neighbors} \\ 0 & \text{otherwise} \end{cases}$$

We see that the first part in Hamiltonian Eq.(6), \hat{H}_0 , is the same as that of a simple lattice under tight-binding approximation (TBA) in solid state physics, so we can rewrite T_{ij} as

$$T_{ij} = N_s^{-1} \sum_k \varepsilon(k) \exp[i\mathbf{k} \cdot (\mathbf{x}_i - \mathbf{x}_j)], \quad (7)$$

where \mathbf{k} is the wave vector in the first Brillouin zone, N_s is the total number of the lattice sites and $\varepsilon(k)$ is the energy spectrum of the Hamiltonian \hat{H}_0 . The inverse transformation is written as

$$\varepsilon(k) = N_s^{-1} \sum_{i,j} T_{ij} \exp[-i\mathbf{k} \cdot (\mathbf{x}_i - \mathbf{x}_j)] \quad (8)$$

and can be approximated by $\varepsilon(k) \approx \varepsilon - \frac{J}{N_s} \sum_{\langle ij \rangle} \exp[-i\mathbf{k} \cdot (\mathbf{x}_i - \mathbf{x}_j)]$ (TBA energy band) in simple cubic lattice. The explicit energy spectrum is seen to be

$$\varepsilon(k) = \varepsilon - Jz \cos\left(\frac{k\lambda}{2}\right) \quad (9)$$

where z is the number of nearest neighbors of each site.

The existence of a finite gap in the excitation spectrum is the characteristic of the MIP. In section II we attempt to determine the SMI phase transition condition from the energy band structure of the ultracold bosonic atoms in optical lattice in terms of Green function method. In section III the Bogliubov transformation is used to obtain exact energy spectrum with which the superfluid phase is explained explicitly.

II. GREEN FUNCTION APPROACH

We begin with the operators, $\hat{a}_i(t)$, $\hat{a}_j^\dagger(t')$, in Heisenberg picture, i.e. $\hat{a}_i(t) = e^{i\hat{H}t}\hat{a}_i e^{-i\hat{H}t}$ and $\hat{a}_j^\dagger(t') = e^{i\hat{H}t'}\hat{a}_j^\dagger e^{-i\hat{H}t'}$ (in the unit of $\hbar = 1$). The retarded single-particle Green function at zero temperature [9] is defined by

$$\begin{aligned}
\left\langle \left\langle \hat{a}_i(t); \hat{a}_j^\dagger(t') \right\rangle \right\rangle &= -i\theta(t-t') \left\langle [\hat{a}_i(t), \hat{a}_j^\dagger(t')] \right\rangle \\
&= -i\theta(t-t') \left\{ \left\langle \hat{a}_i(t) \hat{a}_j^\dagger(t') \right\rangle - \left\langle \hat{a}_j^\dagger(t') \hat{a}_i(t) \right\rangle \right\}
\end{aligned} \tag{10}$$

where $\theta(t-t')$ is the step function:

$$\theta(t-t') = \begin{cases} 1, & t > t' \\ 0, & t < t' \end{cases}$$

The Green function $\left\langle \left\langle \hat{a}_i(t); \hat{a}_j^\dagger(t') \right\rangle \right\rangle$ depends only on the time difference $(t-t')$. The Fourier transformation of the retarded Green function $\left\langle \left\langle \hat{a}_i(t); \hat{a}_j^\dagger(0) \right\rangle \right\rangle$ is seen to be

$$G_{ij}(\omega) \equiv \left\langle \left\langle \hat{a}_i | \hat{a}_j^\dagger \right\rangle \right\rangle_\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \left\langle \left\langle \hat{a}_i(t); \hat{a}_j^\dagger(0) \right\rangle \right\rangle \exp[i(\omega + i\eta)], \quad \eta = +0$$

for a real frequency ω . Using Heisenberg equation, we obtain

$$\omega G_{ij}(\omega) = \left\langle [\hat{a}_i, \hat{a}_j^\dagger] \right\rangle + \left\langle \left\langle [\hat{a}_i, H] | \hat{a}_j^\dagger \right\rangle \right\rangle_\omega \tag{11}$$

which can be evaluated in terms of the commutation relation,

$$[\hat{a}_i, \hat{H}] = \sum_j T_{ij} \hat{a}_j + U \hat{n}_i \hat{a}_i, \tag{12}$$

where $\langle \rangle$ denotes the ground state expectation value. The result is

$$\omega G_{ij}(\omega) = \delta_{ij} + \sum_m T_{im} G_{mj}(\omega) + U \Gamma_{ij}(\omega), \tag{13}$$

where

$$\Gamma_{ij}(\omega) \equiv \left\langle \left\langle \hat{n}_i \hat{a}_i | \hat{a}_j^\dagger \right\rangle \right\rangle_\omega$$

is the higher order Green function which satisfies the following equation

$$\omega \Gamma_{ij}(\omega) = \left\langle [\hat{n}_i \hat{a}_i, \hat{a}_j^\dagger] \right\rangle + \left\langle \left\langle [\hat{n}_i \hat{a}_i, \hat{H}] | \hat{a}_j^\dagger \right\rangle \right\rangle_\omega. \tag{14}$$

It is easy to find

$$[\hat{n}_i \hat{a}_i, \hat{H}] = \varepsilon \hat{n}_i \hat{a}_i + \sum_{j(\neq i)} T_{ij} \hat{n}_i \hat{a}_j + \sum_j T_{ij} \underline{(\hat{a}_i^\dagger \hat{a}_j - \hat{a}_j^\dagger \hat{a}_i)} \hat{a}_i + U \hat{n}_i \hat{n}_i \hat{a}_i \tag{15}$$

Substituting Eq.(15) into Eq.(14), we see that the obtained equation is not closed, because more higher order Green functions appear in the formula of $\Gamma_{ij}(\omega)$. We in this stage use mean field approximation for the underlined operators in Eq.(15), i.e. the number operator \hat{n}_i is replaced by its expectation value $\langle \hat{n}_i \rangle$

$$\sum_j T_{ij} \hat{n}_i \hat{a}_j \approx \sum_j T_{ij} \langle \hat{n}_i \rangle \hat{a}_j \approx n_0 \sum_j T_{ij} \hat{a}_j. \quad (16)$$

In the above approximation we have assumed that the average occupation number of Bose atoms condensed on ground state in each site of the optical lattice is the same, i.e., $\langle \hat{n}_i \rangle \equiv n_0$.

Since the obvious symmetry of $\varepsilon(k) = \varepsilon(-k)$, we have the equality $T_{ij} = T_{ji}$ according to Eq.(8). Utilizing translation symmetry of the Bose system we moreover obtain

$$\sum_j T_{ij} \left(\langle \hat{a}_i^\dagger \hat{a}_j \rangle - \langle \hat{a}_j^\dagger \hat{a}_i \rangle \right) = 0. \quad (17)$$

The Eq. (15) is then simplified as

$$[\hat{n}_i \hat{a}_i, \hat{H}] \approx (\varepsilon + U n_0) \hat{n}_i \hat{a}_i + n_0 \sum_{j(\neq i)} T_{ij} \hat{a}_j.$$

A closed equation for the Green function is seen to be

$$\omega \Gamma_{ij}(\omega) = 2n_0 \delta_{ij} + n_0 \sum_m T_{im} G_{mj}(\omega) + (\varepsilon + U n_0) \Gamma_{ij}(\omega) \quad (18)$$

from which we find

$$\Gamma_{ij}(\omega) = \frac{n_0}{\omega - \varepsilon - U n_0} (2\delta_{ij} + \sum_{m(\neq i)} T_{im} G_{mj}(\omega)). \quad (19)$$

Substitution of $\Gamma_{ij}(\omega)$ in Eq. (19) into (13) yields

$$\omega G_{ij}(\omega) = \delta_{ij} + \sum_m T_{im} G_{mj}(\omega) + \frac{U n_0}{\omega - \varepsilon - U n_0} (2\delta_{ij} + \sum_{m(\neq i)} T_{im} G_{mj}(\omega)) \quad (20)$$

which is the equation for the site space Green function $G_{ij}(\omega)$. The Green function $G_{ij}(\omega)$ is a function of the position difference $(\mathbf{x}_i - \mathbf{x}_j)$ of two sites only for a system with translational invariance. The equation (20) for the Green function $G_{ij}(\omega)$ can be solved with the Fourier

transformation. To this end we express the site space operator \hat{a}_i in terms of the wave-vector-space operator \hat{a}_k

$$\begin{aligned}\hat{a}_i &= \frac{1}{\sqrt{N_s}} \sum_k e^{i\mathbf{k} \cdot \mathbf{x}_i} \hat{a}_k \\ \hat{a}_i^\dagger &= \frac{1}{\sqrt{N_s}} \sum_k e^{-i\mathbf{k} \cdot \mathbf{x}_i} \hat{a}_k^\dagger.\end{aligned}\tag{21}$$

We can prove that the single particle Green function in the Bloch representation is orthogonal, i.e.,

$$\begin{aligned}G_{kk'}(\omega) &\equiv \left\langle \left\langle \hat{a}_k | \hat{a}_{k'}^\dagger \right\rangle \right\rangle_\omega \\ &= \frac{1}{N} \sum_{i,j} e^{-i\mathbf{k} \cdot \mathbf{x}_i} e^{i\mathbf{k}' \cdot \mathbf{x}_j} \left\langle \left\langle \hat{a}_i | \hat{a}_j^\dagger \right\rangle \right\rangle_\omega \\ &= \delta_{kk'} G_k(\omega)\end{aligned}$$

where $G_k(\omega) = \left\langle \left\langle \hat{a}_k | \hat{a}_k^\dagger \right\rangle \right\rangle_\omega$ denotes the orthogonal Green function in Bloch representation.

The Fourier transformation of the Green function $G_{ij}(\omega)$ is then seen to be

$$G_{ij}(\omega) = \frac{1}{N_s} \sum_k e^{i\mathbf{k} \cdot (\mathbf{x}_i - \mathbf{x}_j)} G_k(\omega)\tag{22}$$

Substituting Eq.(22) into Eq.(20) the single particle Green function in the Bloch representation is explicitly obtained as

$$G_k(\omega) = \frac{\omega - \varepsilon + Un_0}{(\omega - \varepsilon(k))(\omega - \varepsilon - Un_0) - Un_0(\varepsilon(k) - \varepsilon)}\tag{23}$$

Rewrite the solution as the following form

$$G_k(\omega) = \frac{A_k^{(1)}}{\omega - E^{(1)}} + \frac{A_k^{(2)}}{\omega - E_k^{(2)}}\tag{24}$$

where $E^{(1)}$ and $E_k^{(2)}$ denote the poles of the Green function $G_k(\omega)$, and it is seen that the excitation spectrum possesses a band structure such as

$$E^{(1)} = \varepsilon, E_k^{(2)} = \varepsilon(k) + Un_0\tag{25}$$

The lowest band shrinks to a single level of zero band width (see Fig.1). Although the energy spectrum Eq.(25) comprises two parts, they may, in a certain case depending on the relative

values of the interatomic repulsion U and the tunnel coupling J , be merged into the same energy band. The energy gap between the two bands is [Fig.1]

$$\Delta = E_k^{(2)}|_{k=0} - E^{(1)} = Un_0 - Jz$$

When the constant of the interatomic repulsion U is large with respect to the tunnel coupling J such that $\Delta > 0$, a gap exists implying the MIP. With increasing of tunnel coupling J the gap width Δ decreases and finally the two energy bands in the excitation spectrum overlap and the gap disappears, indicating the SFP. We then obtain the condition of SMI phase transition that

$$\Delta = 0 \tag{26}$$

,namely,

$$\frac{U}{zJ} = \frac{1}{n_0} \tag{27}$$

which agrees with the result in refs. [7] [8] [10] [11].

To see the SFP more closely we take the zero wave vector limit of the energy band $E_k^{(2)}$ ($k \rightarrow 0$)

$$E_k^{(2)} \sim \varepsilon - Jz + Un_0 + \frac{1}{23}Jz\lambda^2k^2.$$

Under the condition Eq.(27) at which the energy gap between $E^{(1)}$ and $E_k^{(2)}$ disappears, we have a gapless Goldstone mode in the excitation spectrum such as

$$E_{exc} \sim \frac{1}{23}Jz\lambda^2k^2 \tag{28}$$

which is different from the result of Bogoliubov theory for the system of weakly interacting bosons, in the absence of the periodic potential, where the wave vector dependence of the excitation spectrum is linear in the zero wave vector limit so that a non-vanishing velocity can exist. Strictly speaking what we obtained here is an ordinary fluid phase. The energy spectrum of Eq.(25) determined with the help of Green function method is too simple to

realize the superfluid phase explicitly. This may be due to the particular procedure of the approximation used in the above derivation. It is certainly of interest to study the spectrum of bosonic atoms in the BEC trapped in the optical lattice in terms of Bogliubov method to see whether or not the system can possess a superfluid phase which we are going to discuss in the next section.

It is worth while to point out that when the interaction between bosons vanishes *i.e.* $U = 0$, the Green function Eq.(23) reduces to the well known single band solution

$$G_k(\omega) |_{U=0} = \frac{1}{\omega - \varepsilon(k)} \quad (29)$$

for bosons in a periodic potential.

III. BOGLIUBOV METHOD

Now we study the energy spectrum of boson atoms in the optical lattice by means of the Bogliubov method . Using the relation(21) the Hamiltonian(6) can be converted to

$$\hat{H} = \sum_k \varepsilon(k) \hat{a}_k^\dagger \hat{a}_k + \hat{H}_{int}$$

$$\hat{H}_{int} = \frac{U}{2N_s} \sum_{k,p,k',p'} \delta_{k+p,k'+p'} \hat{a}_{k'}^\dagger \hat{a}_{p'}^\dagger \hat{a}_k \hat{a}_p \quad . \quad (30)$$

Since the number of atoms condensed in the zero-momentum state is much larger than one, we have $\hat{a}_0 \hat{a}_0^\dagger = \hat{a}_0^\dagger \hat{a}_0 + 1 \simeq N_0 \gg 1$, where N_0 is the total number of condensed atoms. Thus we can replace the operator \hat{a}_0 and \hat{a}_0^\dagger by a "c" number $\sqrt{N_0}$. The interacting part of the Hamiltonian (30) can be written as (in the order of N_0)

$$\hat{H}_{int} = \frac{U}{2N_s} N_0^2 + \frac{UN_0}{2N_s} \sum_k' (\hat{a}_k \hat{a}_{-k} + \hat{a}_k^\dagger \hat{a}_{-k}^\dagger + 2\hat{a}_k^\dagger \hat{a}_k)$$

and the total Hamiltonian is

$$\hat{H} = \frac{UN_0^2}{2N_s} + N_0 (\varepsilon - zJ) + \sum_k' \left\{ \frac{Un_0}{2} (\hat{a}_k \hat{a}_{-k} + \hat{a}_k^\dagger \hat{a}_{-k}^\dagger) + (\varepsilon(k) + Un_0) \hat{a}_k^\dagger \hat{a}_k \right\} \quad (31)$$

where \sum_k' denotes the sum with exclusion of the term of $k = 0$.

The following Bogoliubov transformation is introduced in order to diagonalize the Hamiltonian(31)

$$\begin{cases} \hat{b}_k = u_k \hat{a}_k + v_k \hat{a}_{-k}^\dagger \\ \hat{b}_k^\dagger = u_k \hat{a}_k^\dagger + v_k \hat{a}_{-k} \end{cases} . \quad (32)$$

We require that the quasi-boson operators \hat{b}_k and \hat{b}_k^\dagger satisfy the usual commutation relation, $[\hat{b}_k, \hat{b}_k^\dagger] = 1$, which leads to the condition

$$u_k^2 - v_k^2 = 1 \quad (33)$$

for the coefficients u_k and v_k and then the Hamiltonian can be written as

$$\hat{H} = E_c + \hat{H}_1 + \hat{H}_2$$

where

$$E_c = \frac{1}{2} U N_0 n_0 + N_0 (\varepsilon - zJ) \quad (34)$$

is a constant and

$$\hat{H}_1 = \sum_k' ((u_k^2 + v_k^2) (\bar{\varepsilon}_k + U n_0) - 2U n_0 u_k v_k) \hat{b}_k^\dagger \hat{b}_k \quad (35)$$

$$\hat{H}_2 = \sum_k' \left(\frac{U n_0}{2} (u_k^2 + v_k^2) - (\bar{\varepsilon}_k + U n_0) u_k v_k \right) (\hat{b}_k \hat{b}_{-k} + \hat{b}_k^\dagger \hat{b}_{-k}^\dagger) \quad (36)$$

To eliminate the off-diagonal part \hat{H}_2 we require

$$\frac{U n_0}{2} (u_k^2 + v_k^2) - (\bar{\varepsilon}_k + U n_0) u_k v_k = 0 \quad (37)$$

where $\bar{\varepsilon}_k = zJ(1 - \cos(kd))$. Introducing a parameter ϕ_k such that

$$u_k = \cosh \phi_k \quad v_k = \sinh \phi_k$$

the conditions (33), (37) lead to the useful relations

$$\tanh 2\phi_k = \frac{2u_kv_k}{u_k^2 + v_k^2} = \frac{Un_0}{\bar{\varepsilon}_k + Un_0}$$

$$u_k^2 + v_k^2 = \cosh(2\phi_k) = \frac{\bar{\varepsilon}_k + Un_0}{E_k}$$

with which the diagonalized Hamiltonian is obtained as

$$\hat{H} = E_c + \sum_k' E_k \hat{b}_k^\dagger \hat{b}_k$$

where the energy spectrum E_k of quasi-particle is

$$E_k = \sqrt{\bar{\varepsilon}_k (\bar{\varepsilon}_k + 2Un_0)}. \quad (38)$$

The energy spectrum is different from that of Eq.(25) and is typical for the superfluid. The energy gap Δ_g of excitation spectrum is obviously

$$\begin{aligned} \Delta_g &= \frac{E_c - N_0\varepsilon}{N_0} \\ &= \frac{1}{2}Un_0 - zJ. \end{aligned} \quad (39)$$

The phase transition condition determined from $\Delta_g = 0$ is

$$\frac{U}{2zJ} = \frac{1}{n_0} \quad (40)$$

which shows a factor 2 difference comparing with the condition in Eq.(27). This may be caused by the approximation itself. When the energy gap disappears, i.e. $\Delta_g = 0$, the dispersion relation of $E_k(k \rightarrow 0)$ reads

$$E_k \sim (zJUn_0d^2)^{1/2} k \quad (41)$$

indicating explicitly the superfluidity in agreement with the Bogliubov superfluid theory for weakly interacting bosons in the absence of the periodic potential. The linear wave vector dependence of the excitation spectrum E_k (unlike the ordinary fluid Eq.(28) where E_{exc} is proportional to k^2) is the characteristic of the sperfluid which gives rise to a persistent velocity of superfluid or quasi-particle found as

$$v_s = \left(\frac{\partial E_k}{\partial k} \right)_{k \rightarrow 0} = (zJU n_0 d^2)^{1/2}. \quad (42)$$

For the case of boson atoms with repulsive interaction ($a_s > 0$), the parameters J and U are positive, v_s is a real number which implies a persistent current. The velocity v_s can be controlled by the tuning of laser lights which result in the optical lattice. As seen from the definitions (4), (5) for J and U , these parameters both depend on the Wannier functions which are essentially determined by the potential of optical lattice. Therefore J and U are not independently tunable by the adjusting of the laser parameters. In fact when the depth of the lattice potential increases, the hopping matrix element J decreases exponentially while the matrix- element of the on-site interaction, U , increases. We thereby expect that there exists a maximum value of the persistent velocity v_s in some particular values of J and U .

IV. CONCLUSION

We have studied the Bose-Hubbard model of BEC trapped in a periodic potential in terms of Green function method and Bogliubov transformation as well. The condition of phase transition between SFP and MIP is determined by the energy band structure of the excitation spectrum due to, obviously, the competition between the interatomic repulsion and the tunnel coupling. Our result agrees with the condition of SMI phase transition obtained in literature. The SFP property of BEC in the optical lattice is explained explicitly from energy spectrum derived by means of the Bogliubov approach. It is shown that the persistent velocity of the quasi-particle in SFP can be tuned by the adjusting of the laser lights which result in the optical lattice.

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Figure captions

Fig. 1. Excitation spectrum and energy gap.

